Mathematical Modelling and Evaluation of the Islamic Derivative Arbun

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Abstract

Options are derivatives, meaning they derive their value from an underlying financial instrument. Though options can be entered using stock as the underlying security, indexes and futures also have options available. Options are an extremely versatile instrument and can be used to create a variety of different limited risk strategies.

In Islamic finance, these products are not used because their principles are not consistent with the principles of the Islamic Sharia’a.

However, there are derivatives that are equivalent to the latter and which may be subject to assets risk management. Among these Islamic products we have “Arbun”, this one not often used in reality because of the difficulty of its evaluation as there is no mathematical model that can describe it.

In this context and in order to overcome this issue, we present in this paper a new mathematical approach to model the derivative “Arbun” based on stochastic processes and genetic algorithms.

The main purpose of this approach is to offer a decision tool to investors to control the market risk in Islamic finance.

Keywords: Derivatives, Classical Finance, Islamic Finance, Islamic Sharia’a, Mathematical Modelling, Arbun, Stochastic Process, Genetic Algorithms.
1. Introduction

A derivative is a financial instrument whose value depends on the value of another financial asset called the underlying and that can be a stock, bond, currency, etc.

Derivatives or conventional derivatives are the most important financial instruments in the financial field, so their role becomes very important in the financial market because of technological progress and financial development, as well as significant variations noticed among financial assets, exchange rate, interest rates, etc.

Derivatives have expanded considerably in recent years which has led to innovation of new tools aiming transfer, diversification, hedging to protect investments against financial risks, speculation and access of investment in the financial market in case of insufficient liquidity.

Despite the importance of derivatives in the financial markets they are not used in Islamic finance because their principles are illegal according to their principle.

Despite the importance of derivatives in the financial markets, they are not used in Islamic finance because their principle is deemed to be illegal.

However, in Islamic finance there are derivatives that are equivalent and that can lead to access to investments in portfolios of financial assets and to hedge and risk management of these portfolios.

Among the most popular products in Islamic finance is the Arbun. This product is still not used in practice because we cannot evaluate it as there is no model that can describe it.

To overcome this problem and provide investors in Islamic finance a decision tool to hedge against market risk, we present in this paper a new mathematical modelling approach of the derivative Arbun based on stochastic processes and genetic algorithms.

In Section 1, we present the classical derivatives, the basic principles of Islamic finance like the prohibition of conventional derivatives are described in Section 2, then the Islamic derivative Arbun will be detailed in Section 3. In section 4, the mathematical modelling of the Islamic derivative will be developed.

2. Conventional derivatives

Conventional derivative is a tradable contract between two speakers, one is called buyer and the other is seller called, and whose value depends on the value of another financial asset called the underlying and that can be a stock, bond, currency, etc.

In the derivatives market, there are many types of derivatives such as forward contracts, futures, swaps and options that are most used in conventional finance.

3. Principles of the Islamic finance

Let us briefly recall basic principles of Islamic finance

a- The prohibition of the practice of interest (Riba)

This means that in Islam, it is forbidden that money generates itself money. This prohibition can be explained by the fact that money has no intrinsic value; it is only an exchange instrument.

b- The sharing of profits and losses

This principle is used to relate the benefit to the risk; it states that the fund investor assumes all liability associated with his partners and also the sharing of profits and losses between them in all financial operations.

c- The prohibition of speculation (Gharar)
This is a contract on uncertain elements or which are not clearly defined.

d- **The prohibition of gambling (The Maysir and Qimar)**

It is a kind of bet that is made due to the occurrence of certain events based on subjective estimates.

e- **The prohibition of illicit economic activities according to Shariah**

This means that every trader in Islamic finance should opt only for activities that are authorized by the Shariah, and avoid all other prohibited things such as alcohol or pork.

4. **Modelling the product Arbun**

4.1. **The Islamic derivative Arbun**

Let consider a contract of buying a product $P$ between a buyer and a seller whose price is equal to $X$. The buyer can pay a proportion of the amount $X$ noted “$u$” at the beginning of the contract to the seller as the rest “$v$” of the amount $X$ will be paid at the end of the period needed to get the product.

The value of this proportion and the period are negotiated between the buyer and seller. At maturity, if the buyer exercised this contract it pays only the value $v$ to get the product, however if he abandons the contract he loses the value $u$.

The proportion “$u$” is called Arbun and the contract is called Arbun contract.

The principle of Arbun contract is close to that of the option except in the case of Arbun contract, the Arbun is a part of the amount of the product while in the case of option contract there is an additional bonus to be paid.

4.2. **Hypothesis**

a) A perfect securities market (that is: no taxes; no transaction costs; no restrictions on selling short; all information is free and simultaneously available to all investors).

b) At time $t = 0$, there is a negotiation between buyer and seller on Arbun and giving rise to a minimum threshold proposed by the purchaser and a maximum of Arbun proposed by the seller.

c) At maturity $t = T$, the buyer has the right to exercise or not the contract of purchase and the seller remains at the disposal of the buyer.

d) No short sale restrictions.

e) The price of the underlying asset follows a geometric Brownian motion process and the value of the Arbun is part of the underlying value.

Under these assumptions, the buyer has the right to exercise or not Arbun contract. His decision is based on the value of the underlying at maturity. For this, he proposes a minimum value of Arbun, noted $Arbun_{\text{min}}$ because this one will be lost in case of non-execution of the contract Arbun and if the underlying value has decreased at maturity.

Therefore, Arbun is part of an interval whose lower limit is equal $Arbun_{\text{min}}$ and the upper bound is equal $Arbun_{\text{max}}$.

Basically, before exercising the contract Arbun, the buyer makes the comparison between the value of the underlying at maturity $S_T$ and that of the initial time $S_0$.

If $S_T > S_0$ the buyer exercises the underlying contract Arbun else he abandons. In case if the buyer abandons the contract Arbun, the seller loses an amount whose value is $S_0 - S_T$, this means that the
buyer has contributed negatively in creating a situation in which the seller has missed an opportunity to sell the underlying with a price $S_0$. This leads the seller to consider this situation and requests from the buyer a value of Arbun that is equals to $S_T - S_0$.

So it takes to calculate the amount $E(S < S_0) = E(S - S_0)^-$ for the benefit of the seller because he missed this opportunity which is due by the buyer.

As there is a negotiation between buyer and seller on Arbun, the buyer seeks to $\min\{E(S - S_0)\}$ and the seller seeks to $\max\{E(S - S_0)\}$.

4.3. Mathematical Modelling of Arbun

The price of a commercial product $S_t$ at time $t$ is a random variable whose evolution over time can be modelled by a stochastic process $S = (S_t, t \geq 0)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ the Black & Scholes differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dz$$  \hspace{1cm} (1)

It is well-known that the stochastic process defined by relation (1) is a Black-Scholes-Samuelson process (see for example Janssen, Manca and Volpe (2009), chapter 14) for which

- The constant drift $\mu$ indicates the expected return of the commercial product price per unit time;
- $\sigma$ is a constant indicating the annual volatility of the commercial product price. From classical results on this process, we have :

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma z(t)}$$  \hspace{1cm} (2)

Some assumptions of the Black & Scholes model such as The AOA and the existence of constant and positive free interest risk are no longer valid in our Arbun valuation model because of some restrictions in Islamic Finance.

we have $dS_t = \mu S_t dt + \sigma S_t dz$ so $\frac{dS_t}{S_t} = \mu dt + \sigma dz$

we have $E(S - S_0)^- = E(S - S_0) - E(S - S_0)^+$

$$E(S - S_0)^- = \int_{-\infty}^{+\infty} (S - S_0) f(S) ds - \int_{S_0}^{+\infty} (S - S_0) f(S) ds$$

as $f(S) = \frac{1}{S \sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln(S) - \mu}{\sigma}\right)^2}$ so

$$E(S - S_0)^- = \int_{-\infty}^{+\infty} (S - S_0) \times \frac{1}{S \sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln(S) - \mu}{\sigma}\right)^2} ds - \int_{S_0}^{+\infty} (S - S_0) \times \frac{1}{S \sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{\ln(S) - \mu}{\sigma}\right)^2} ds$$

Let $z = \frac{\ln(S) - \mu}{\sigma} \Rightarrow dz = \frac{1}{S} ds \Rightarrow ds = Sdz$ and $S = e^{\sigma z+\mu}$
\[
E(S-S_0) = \int_0^{\infty} \left( e^{z^2/2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right) dz - \int_0^{\infty} \left( e^{z^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2} \right) dz
\]

\[
E(S-S_0) = \left[ \int_0^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} dz - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right] - \int_0^{\infty} \left( \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} \right) dz
\]

\[
E(S-S_0) = \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right]
\]

\[
S_0 \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right]
\]

\[
E(S-S_0) = \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right]
\]

\[
\text{as } F(x) = \int_0^x f(t) dt \quad \text{where } f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}
\]

\[
E(S-S_0) = e^{-\mu^2/2} \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right] - S_0 \left[ F\left( \frac{\mu}{\sigma} \right) - F\left( -\ln(S_0) + \mu \right) \right]
\]

Let \( t = -(z-\sigma) \Rightarrow dz = -dt \)

\[
E(S-S_0) = e^{-\mu^2/2} \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-z^2/\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \right] - S_0 \left[ F\left( \frac{\mu}{\sigma} \right) - F\left( -\ln(S_0) + \mu \right) \right]
\]

\[
E(S-S_0) = e^{-\mu^2/2} \left[ F\left( \frac{\mu}{\sigma} \right) - F\left( -\ln(S_0) + \mu \right) \right] + S_0 \left[ F\left( -\ln(S_0) + \mu \right) - F\left( \frac{\mu}{\sigma} \right) \right]
\]

Let \( a(\mu,\sigma) = -\ln(S_0) + \mu \), \( b(\mu,\sigma) = -\ln(S_0) + \mu \), \( c(\mu,\sigma) = \frac{\mu}{\sigma} \), \( d(\mu,\sigma) = e^{\mu^2/2} \)

\[
e(\mu,\sigma) = \frac{\mu}{\sigma} + \sigma
\]

\[
E(S-S_0) = d(\mu,\sigma) \times \left( F[e(\mu,\sigma)] - F[b(\mu,\sigma)] \right) + S_0 \times \left( F[a(\mu,\sigma)] - F[c(\mu,\sigma)] \right)
\]

(3)

Let \( (\mu_{\text{min}}, \sigma_{\text{min}}) \) and \( (\mu_{\text{max}}, \sigma_{\text{max}}) \) two values of \( (\mu,\sigma) \) which respectively minimize and maximize the amount of \( E(S-S_0) \).

These values are obtained by an optimization procedure using genetic algorithms subject to many constraints. So:

\[
Arbun_{\text{max}} = d(\mu_{\text{max}}, \sigma_{\text{max}}) \times \left( F[e(\mu_{\text{max}}, \sigma_{\text{max}})] - F[b(\mu_{\text{max}}, \sigma_{\text{max}})] \right) + S_0 \times \left( F[a(\mu_{\text{max}}, \sigma_{\text{max}})] - F[c(\mu_{\text{max}}, \sigma_{\text{max}})] \right)
\]
\[ Arbun_{\min} = d(\mu_{\min}, \sigma_{\min}) \times (F[c(\mu_{\min}, \sigma_{\min})] - F[b(\mu_{\min}, \sigma_{\min})]) + S_0 \times (F[a(\mu_{\min}, \sigma_{\min})] - F[c(\mu_{\min}, \sigma_{\min})]) \]

Then

\[ Arbun \in ]Arbun_{\min}, Arbun_{\max}[ \]

The optimization (maximization-minimization) of the function \( g \) (defined below) will be done by genetic algorithm.

Let us briefly recall that the concept of a genetic algorithm was originally developed by John Holland [7]. It has an iterative form for finding optimum and manipulate a population of constant size. This population is composed of candidate points called chromosomes. The constant size of the population leads to a phenomenon of competition between chromosomes. Each chromosome represents the encoding of a potential solution to the problem to be solved, which can take several values belonging to an alphabet which is not necessarily digital [8]. The various operations involved in a basic genetic algorithm are:

**a- Initialization**

The population is a set of chromosomes which are composed of genes representing the values of \( x \) vector, which \( x \) is the value of the couple \( (\mu, \sigma) \).

This population is initially randomly using real code.

**b- Evaluation Function**

The following operation is the evaluation of chromosomes generated by the previous operation by an evaluation function (fitness function), while the design of this function is a crucial point in using GA. The fitness function used in this work is:

\[ g(\mu, \sigma) = d(\mu, \sigma) \times (F[e(\mu, \sigma)] - F[b(\mu, \sigma)]) + S_0 \times (F[a(\mu, \sigma)] - F[c(\mu, \sigma)]) \]

**c- Operations of selection**

After the operation of the assessment of the population, the best chromosomes are selected using the wheel selection that is associated with each chromosome a probability of selection, noted, \( P_i \).

\[ P_i = \frac{1}{N-1} \left( 1 - \frac{f_i}{\sum_{i \in Pop} f_i} \right) \]  \hspace{1cm} (4)

Some chromosomes will be "more" reproduced and other "bad" will be eliminated.

**d- Operations crossing**

After using the selection method for the selection of two individuals, we apply the Crossover operator to a point on this couple. This operator divides each parent into two parts at the same position, chosen randomly. The child 1 is made of a part of the first parent and the second part of the second parent when the child 2 is composed of the second part of the first parent and the first part of the second parent.

**e- Operation of mutation**

This operation gives to genetic algorithms property of ergodicity [7] which indicates that it will be likely to reach all parts of the state-owned space, without the travel all in the resolution process. This is usually to draw a random gene in the chromosome and replace it with a random value.
f- Conditions for Convergence

At this level, the final generation is considered. If the result is favorable then the optimum chromosome is obtained. Otherwise the evaluation and reproduction steps are repeated until a certain number of generations, until a defined or until a convergence criterion of the population are reached. The results obtained are the proportions \( x \) are the desired optimal solution that maximizes (respectively minimizes) the function \( f(x) \), ie: \((\mu_{\text{max}}, \sigma_{\text{max}})\) (respectively \((\mu_{\text{min}}, \sigma_{\text{min}})\)).

\[
[Arbun_{\text{Min}}, Arbun_{\text{Max}}] = \text{Optimization Procedure } g(\mu, \sigma)
\]

\[
Arbun_{\text{Max}} = \text{GA Max } g(\mu, \sigma)
\]

\[
Arbun_{\text{Min}} = \text{GA Min } g(\mu, \sigma)
\]

S.C :
\[
\mu \geq \mu^*
\]
\[
\sigma \leq \sigma^*
\]

\[
Arbun_{\text{Max}} < S_0
\]

\[
Arbun_{\text{Min}} > 0
\]

\[
Arbun_{\text{Min}} < Arbun_{\text{Max}}
\]

Figure 1: Calculation algorithm Arbun

The values of \(\mu^*\) and \(\sigma^*\) are initialized and defined as follows:

\[
\mu^* = \frac{1}{T} \sum_{i=1}^{T} \ln \left( \frac{S_i}{S_{i-1}} \right) = \frac{1}{T} \sum_{i=1}^{T} u_i \quad \text{and} \quad \sigma^* = \sqrt{\frac{1}{T-1} \sum_{i=1}^{T} (u_i - u^*)^2}
\]

Where \(S_i\) is the price of the underlying at time \(i\).

5. Numeric Application

We present in this section, a digital implementation of the algorithm for calculating the range of the Ar bun.

Consider a set of financials assets that have initials values \((S_{0,i}^1, S_{0,i}^2, \ldots, S_{0,i}^{10})\) and it has historical data. This table shows the lower bound (Ar bun_{\text{Min}}) and the upper bound (Ar bun_{\text{Max}}) for each value of the financial asset for a given period.

<table>
<thead>
<tr>
<th>(S_{0,i}^i) ((i = 1, \ldots, 10)) mile DH</th>
<th>Arbun_{\text{Min}} mile DH</th>
<th>Arbun_{\text{Max}} mile DH</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0.04195</td>
<td>2.2326</td>
</tr>
<tr>
<td>3.5</td>
<td>0.4888</td>
<td>2.7908</td>
</tr>
<tr>
<td>4</td>
<td>1.0422</td>
<td>3.3490</td>
</tr>
<tr>
<td>4.5</td>
<td>1.6992</td>
<td>3.9071</td>
</tr>
<tr>
<td>5</td>
<td>2.4668</td>
<td>4.4653</td>
</tr>
<tr>
<td>5.5</td>
<td>3.3506</td>
<td>5.0235</td>
</tr>
<tr>
<td>6</td>
<td>4.2974</td>
<td>5.5812</td>
</tr>
<tr>
<td>6.5</td>
<td>5.4194</td>
<td>6.1397</td>
</tr>
<tr>
<td>7</td>
<td>6.55485</td>
<td>6.6980</td>
</tr>
<tr>
<td>7.5</td>
<td>7.2867</td>
<td>7.2561</td>
</tr>
</tbody>
</table>

Table 1: Arbun_{\text{Min}} and Arbun_{\text{Max}} of financial assets (mile DH) for a given period
6. Conclusion

In this paper, we presented conventional derivatives and the principles of Islamic finance, which considers these products as illegal according to the principles of the Sharia'a. Thus we presented the Islamic derivative called Arbun and described its features that differentiate it from the option.

In addition, we have developed a new mathematical modelling approach to Islamic derivative Arbun using stochastic processes and genetic algorithms.

This approach is considered as an alternative approach to the option which is a conventional derivative considered as illegal according the principle of Islamic finance. The new approach allows investors to hedge against market risk and formed a decision tool decision to them to protect their patrimony.

References


النمذجة الرياضية وتقييم المشتق الإسلامي أربون

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الملخص:

الخيارات هي المشتقات، بمعنى أنها تستمد قيمتها من أداة مالية أساسية، على الرغم من الخيارات يمكن إدخالها باستخدام الأسهم كما الأمانة الأساسية، والمؤشرات والعقود الآجلة أيضا خيارات المتاحة. الخيارات هي أداة متعددة الاستخدامات، ويمكن استخدامها لخلق مجموعة متنوعة من استراتيجيات المخاطر المحدودة المختلفة.

وفي مجال التمويل الإسلامي، لا تستخدم هذه المنتجات لأن مبادئها لا تتفق مع مبادئ الشريعة الإسلامية. ومع ذلك، هناك مشتقات تعادل القيمة الأخيرة والتي قد تخضع لإدارة مخاطر الأصول. من بين هذه المنتجات الإسلامية لدينا "أربون"، وهذا واحد لا تستخدم في كثير من الأحيان في الواقع بسبب صعوبة تقييمه حيث لا يوجد نموذج رياضي يمكن وصف ذلك.

في هذا السياق ومن أجل التغلب على هذه المسألة، نقوم في هذه الورقة نهجًا رياضيًا جديدًا لنموذج مشتق "أربون" على أساس العمليات العشوائية والخوارزميات الجينية.

والغرض الرئيسي من هذا النهج هو تقديم أداة قرار للمستثمرين للسيطرة على مخاطر السوق في التمويل الإسلامي.

الكلمات المفتاحية: المشتقات، التمويل الكلاسيكي، التمويل الإسلامي، الشريعة الإسلامية، النمذجة الرياضية، أربون، العملية العشوائية، الخوارزميات الجينية.